**Modeling Stability in the Mathematical Principles of Reinforcement from Dynamic Equations to Tensor Visualization**

This modeling workflow began with a central question: How can we identify the equilibrium conditions and stability boundaries of a reinforcement learning system rooted in the Mathematical Principles of Reinforcement (MPR)? To answer this, we:

* Formalized a set of dynamic equations capturing MPR behavior
* Simulated and analyzed the equilibria and attractor basins
* Expanded the system to include coupling (creating a 3D dynamic system)
* Performed a tensor-based parameter sweep across key variables
* Fitted a logistic model to estimate the boundary of stability
* Built interactive visualizations for scientific exploration

**Foundational Equations and Equilibrium Conditions:**

We began with a simplified 2D MPR system where *b =* response rate*, r* = reinforcement rate, and *a* = arousal (Killeen, 2023). As equations, we have:

Response equation:

Reinforcement rate as a function of responding:

Arousal modeled dynamically:

To find the **equilibrium**, we solved the system:

Setting the two expressions for *b*∗ to be equal gives us:

Solving this cubic equation revealed two attractors: a trivial one at zero, and a non-trivial equilibrium at higher values of *a, b,* and *r* that were dependent on initial conditions.

**Extending to a 3D System with Coupling:**

To reflect **temporal contiguity** and operant-reinforcer binding, we introduced a third variable, **coupling** ccc, updated dynamically:

We replaced the 2D response equation with:

This resulted in a 3D autonomous system:

We visualized the vector field and phase trajectories in 3D, confirming the model’s bistable nature and stable attractors.

**Simulation, Bifurcation Analysis, and Tensor Representation:**

We conducted:

* A **bifurcation analysis** varying ​ to identify regions of behavioral collapse vs. persistence.
* A **vector field analysis** of arousal–response–coupling dynamics.
* A **parameter sweep** across α\alphaα and β\betaβ (sensitivity and coupling), repeated at 10 levels of reinforcer magnitude mmm.

This produced a **3D tensor**:

From the tensor, we extracted:

* **Stability classification** (binary threshold at b∗>1.0b^\* > 1.0b∗>1.0)
* **Contours** where behavior transitions between collapse and persistence
* **Heatmaps** and **animated surfaces** of system behavior

**Logistic Model Fitting and Boundary Analysis:**

We trained a logistic regression model:

This provided:

* Smooth surface predictions of stability probability
* Fast approximations for decision support
* Good alignment with the **true boundaries** extracted from simulation

We plotted the predicted surface alongside the extracted contours for comparison.

**Conclusion:**

This entire modeling framework offers a replicable and interpretable approach to:

* Quantifying conditions for sustained operant behavior
* Visualizing response surfaces
* Simulating verbal or nonverbal reinforcement dynamics over time